Student Notes

Prep Session Topic: Area and Volume Problems.

Except for the 2009 BC exam, an area and volume problem has appeared on every one of the free-response sections of the AP Calculus exam AB and BC since year 1 (= 1969). They are straightforward and only occasionally have anything out of the ordinary.

You may be asked to find the point(s) of intersection of the given functions in order to use them as the limits of integration. Show the equation you are solving, not just the solution. Finding the values is not enough: you do not earn the point until you use the value correctly as a limit of integration somewhere in the problem.

This question will probably appear in the no calculator section. This means that any integration or equation solving involved will be relatively easy. Before 2009 this question was almost always in the calculator allowed section. Therefore, if you are practicing with some old problems, you may need a calculator. Concentrate on setting up the integrals needed; this does not change if you have ort do not have a calculator. (See the note on page 2 for calculator notes, just in case.)

Simple questions may also appear on the multiple-choice section.

What should you know how to do?

- Solve equations. Remember to show the equation you are solving.
- Graph functions. Often the graph is given; occasionally you will want to sketch your own graph to be sure which curve is "on top".
- Find the area of a region enclosed by two functions over an interval. You may have to find the endpoints of the interval to use as limits of integration.
- Find the volume when the region is revolved around an axis or a line using the "washer" or "disk method". If you learned the "method of cylindrical shells", you may use it; however, any volume you are asked for can be done by the washer/disk method.
- Find volumes of solid figures with regular cross sections – squares, equilateral triangles, semi-circles, etc.

Rounding & Simplifying

Some parts may ask you to, "Write, but do not evaluate, an integral expression.…" In other words you are to set up the integral for the area or volume, but you do not have to find an antiderivative or the numerical answer.

If you are asked to find the numerical answer, remember <u>Answers need not be simplified either numerically or algebraically</u>, so DO NOT SIMPLIFY. If you get 1 + 1 leave it! A correct answer simplified incorrectly will lose the point. Do not try to change your answer to a decimal.

Student Notes

Multiple-choice questions:

Underlined questions are calculator required; others are no calculator allowed

Area: 1, 2, <u>3</u>, <u>4</u>, <u>5</u>
Volume of revolution: 6,
Volume by slicing: <u>7</u>, 8, <u>9</u>, <u>10</u>, <u>11</u>

Answers: 1 D, 2 D, 3 B, 4 C, 5 D, 6 A, 7 C, 8 B, 9 D, 10 C, 11 B.

WATCH and LISTEN to the multiple-choice questions being solved

Go to http://tinyurl.com/NMSI-Math-4 Click on the "Full Screen" arrow. Then click anywhere on the page to see and hear from that point on.

Click anywhere to go back anytime.

Free-response questions included are:

2009 AB 4, 2009 AB 4 form B, D&S Review book question (used with permission)

<u>If this question appears on the calculator allowed section</u>, keep the following in mind:

If you are asked to find the point(s) of intersection of the given functions in order to use them as the limits of integration. Show the equation you are solving, not just the solution. You may use your calculator to find the values.

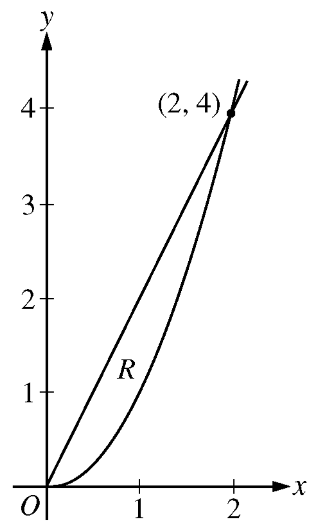
You are expected to use your calculator to evaluate the definite integrals you set up. Remember to show the integral with limits of integration you are evaluating in standard notation. On the calculator allowed section you do not need to show an antiderivative (and an incorrect antiderivative could cost you a point).

The test directions say, "Your final answers must be accurate to three places after the decimal point." <u>This is what "accurate to three places after the decimal point" means</u>: the first three digits after the decimal point must be correct by truncating or rounding. However, you are not required to round or truncate to three-places. Anything written after the third decimal place, right or wrong, is ignored. Thus any of these values for π (which should be left as π) is acceptable: 3.1415926…, 3.141, 3.142, 3.1416, 3.141789…, 3.1425643…. Once again: DON'T ROUND, a correct answer that is then rounded incorrectly loses the answer point.

CALCULUS AB
SECTION II, Part B
Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



4. Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.

   (a) Find the area of R.

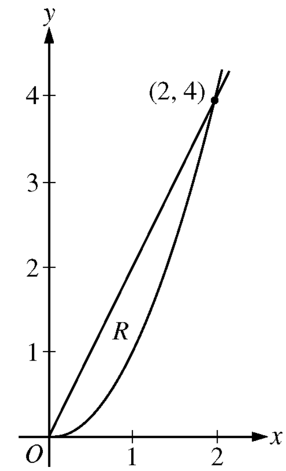
   (b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

   (c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

WRITE ALL WORK IN THE PINK EXAM BOOKLET.

Question 4

Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

(c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

(a) Area $= \int_0^2 \left(2x - x^2\right) dx$

$= x^2 - \frac{1}{3}x^3 \Big|_{x=0}^{x=2}$

$= \frac{4}{3}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \int_0^2 \sin\left(\frac{\pi}{2}x\right) dx$

$= -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big|_{x=0}^{x=2}$

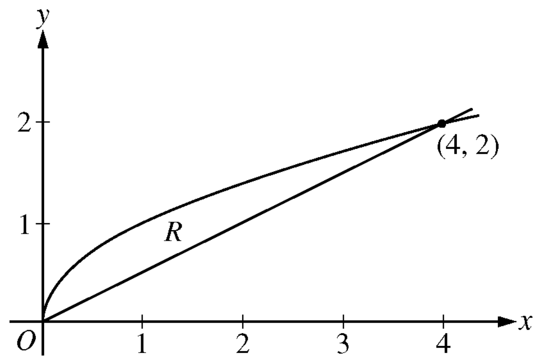
$= \frac{4}{\pi}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right)^2 dy$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \end{cases}$

Question 4



Let R be the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$, as shown in the figure above.

(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are squares. Find the volume of this solid.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 2$.

(a) Area $= \displaystyle\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{2}{3}x^{3/2} - \frac{x^2}{4}\Bigg|_{x=0}^{x=4} = \frac{4}{3}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

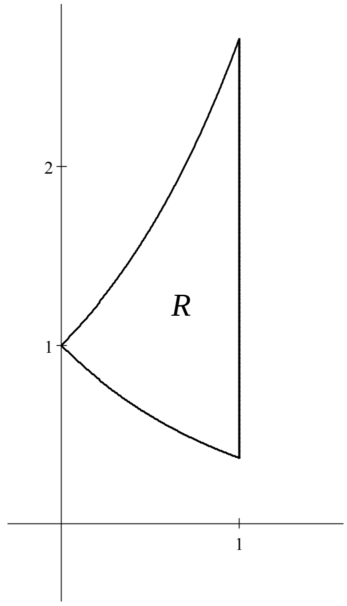
(b) Volume $= \displaystyle\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx = \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4}\right) dx$

$$= \frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x^3}{12}\Bigg|_{x=0}^{x=4} = \frac{8}{15}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \pi \displaystyle\int_0^4 \left(\left(2 - \frac{x}{2}\right)^2 - \left(2 - \sqrt{x}\right)^2\right) dx$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \end{cases}$



Let R be the region bounded by the graphs of $y = e^{x}$, $y = e^{-x}$ and the vertical line $x = 1$ as shown in the figure above.

(a) Find the area of R
(b) Find the volume of the solid generated when R is revolved around the x-axis.
(c) The region R is the base of a solid. For each $x, 0 \le x \le 1$, the cross section perpendicular to the x-axis will be rectangles. The base of these rectangles lies in the region R and the heights of the rectangles are equal to the x-coordinate of the cross section. Write, but do not evaluate, an integral expression that gives the volume of the solid.

Answers:

(a) $e + e^{-1} - \left(e^{0} + e^{0}\right)$ or $e + e^{-1} - 2$

(b) $\pi\left(\dfrac{e^{2}}{2} + \dfrac{e^{-2}}{2} - \left(\dfrac{1}{2} + \dfrac{1}{2}\right)\right)$

(c) $\int_0^1 x\left(e^{x} - e^{-x}\right)dx$

From *Multiple-choice and Free-response Questions in Preparation for the AP Calculus (AB) Examination* D&S Marketing Systems, Inc. Brooklyn, NY

Name: ________________________ Class: ___________________ Date: __________ ID: A

Area - Volume Problems

Multiple Choice
Identify the choice that best completes the statement or answers the question.

Underlined questions are calculator required; others are no calculator.

Area: 1, 2, 3, <u>4</u>, <u>5</u>
Volume of revolution: 6,
Volume by slicing: 7, 8, <u>9</u>, <u>10</u>, <u>11</u>

1. The area of the region enclosed by the graph of $y = x^2 + 1$ and the line $y = 5$ is

a. $\frac{14}{3}$
b. $\frac{16}{3}$
c. $\frac{28}{3}$
d. $\frac{32}{3}$
e. 8π

2. What is the area of the region between the graphs of $y = x^2$ and $y = -x$ from $x = 0$ to $x = 2$?

a. $\frac{2}{3}$
b. $\frac{8}{3}$
c. 4
d. $\frac{14}{3}$
e. $\frac{16}{3}$

3. Let R be the region enclosed by the graph of $y = 1 + \ln\left(\cos^4 x\right)$ the x-axis, and the lines $x = -\frac{2}{3}$ and $x = \frac{2}{3}$. The closest integer approximation of the area of R is

a. 0
b. 1
c. 2
d. 3
e. 4

4. What is the area of the region in the first quadrant enclosed by the graphs of $y = \cos x$, $y = x$, and the y-axis?

a. 0.127
b. 0.385
c. 0.400
d. 0.600
e. 0.947

5. If $0 \le k < \frac{\pi}{2}$ and the area under the curve $y = \cos x$ from $x = k$ to $x = \frac{\pi}{2}$ is 0.1, then $k =$

a. 1.471
b. 1.414
c. 1.277
d. 1.120
e. 0.436

6. If the region enclosed by the y-axis, the line $y = 2$, and the curve $y = \sqrt{x}$ is revolved about the y-axis, the volume of the solid generated is

a. $\frac{32\pi}{5}$
b. $\frac{16\pi}{3}$
c. $\frac{16\pi}{5}$
d. $\frac{8\pi}{3}$
e. π

7. The base of a solid S is the region enclosed by the graph of $y = \sqrt{\ln x}$, the line $x = e$, and the x-axis. If the cross sections of S perpendicular to the x-axis are squares, then the volume of S is

a. $\frac{1}{2}$
b. $\frac{2}{3}$
c. 1
d. 2
e. $\frac{1}{3}\left(e^3 - 1\right)$

8. The base of a solid is the region in the first quadrant enclosed by the graph of $y = 2 - x^2$ and the coordinate axes. If every cross section of the solid perpendicular to the y-axis is a square, the volume of the solid is given by

a. $\pi\int_0^2 (2-y)^2 dy$

b. $\int_0^2 (2-y) dy$

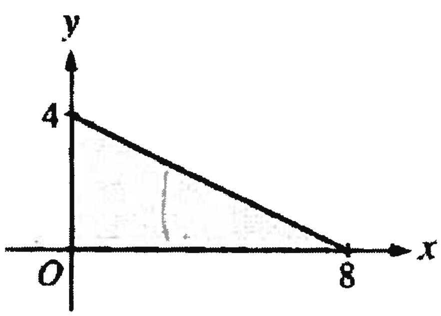
c. $\pi\int_0^{\sqrt{2}} (2-x^2)^2 dx$

d. $\int_0^{\sqrt{2}} (2-x^2)^2 dx$

e. $\int_0^{\sqrt{2}} (2-x^2) dx$

9. The region bounded by the graph of $y = 2x - x^2$ and the x-axis is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an equilateral triangle. What is the volume of the solid?

a. 1.333
b. 1.067
c. 0.577
d. 0.462
e. 0.267



10.

The base of a solid is a region in the first quadrant bounded by the x-axis, the y-axis, and the line $x + 2y = 8$, as shown in the figure above. If cross sections of the solid perpendicular to the x-axis are semicircles, what is the volume of the solid?

a. 12.566
b. 14.661
c. 16.755
d. 67.021
e. 134.041

11. The base of a solid is the region in the first quadrant bounded by the y-axis, the graph of $y = \tan^{-1} x$, the horizontal line $y = 3$, and the vertical line $x = 1$. For this solid, each cross section perpendicular to the x-axis is a square. What is the volume of the solid?

a. 2.561
b. 6.612
c. 8.046
d. 8.755
e. 20.773